Multirate explicit stabilized methods based on a modified equation for problems with multiple scales

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SIAM CSE21

Problem statement

Multirate ODE

Solve the stiff system

$$y' = f(y) := f_F(y) + f_S(y),$$
$$y(0) = y_0.$$

Term	Stiff ?	Cost ?
f_F	stiff	cheap
f_S	nonstiff	expensive
$f_F + f_S$	stiff	expensive

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- Common methods are explicit or implicit, both with their advantages and disadvantages.
- But there's also a family in between: *explicit stabilized methods*.
  - fully explicit,
  - stability domain grows quadratically with function evaluations,
  - no step size restrictions.

Very efficient for large stiff nonlinear problems.

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RKC (Van der Houwen and Sommeijer, 1980)

- τ the step size, ρ spectral radius of $\partial f/\partial y$,
- $s \in \mathbb{N}$ such that $\tau\rho \leq 2s^2$.

$$k_0 = y_n, \qquad k_1 = k_0 + \mu_1 \tau f(k_0),$$
$$k_j = \nu_j k_{j-1} - \kappa_j k_{j-2} + \mu_j \tau f(k_{j-1}) \quad \text{for } j = 2, \ldots, s,$$
$$y_{n+1} = k_s.$$

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Our goals

Goals:

- Design a multirate RKC (*mRKC*) method and recover the efficiency of RKC (Abdulle et al., 2020) (today).
- Use a methodology which can be extended to multirate SDEs (Abdulle and Rosilho de Souza, 2020) (not today).

Spoiler on mRKC:

- fully explicit,
- no need for scale separation,
- stable on a large region along the negative real axis,
- first-order accurate (we aim at SDEs, not high order).

Modified multirate equation

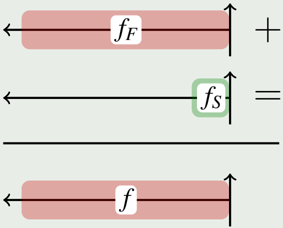
Idea

Shrink the spectrum of f_F and integrate a modified equation.

Original problem

$$y' = f(y) = f_F(y) + f_S(y).$$

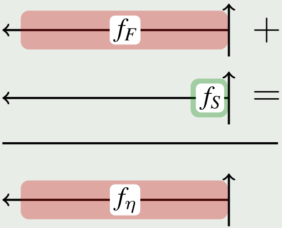
Spectral properties:



Modified equation

$$y_\eta' = f_\eta(y_\eta) \quad \text{with } \eta \geq 0.$$

For $\eta = 0$ it holds $f_\eta = f$ hence:



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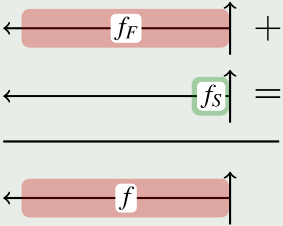

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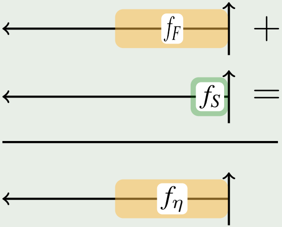
$$y'_\eta = f_\eta(y_\eta) \quad \text{with } \eta \geq 0.$$

For $\eta > 0$ then $f_\eta = f + \mathcal{O}(\eta)$,



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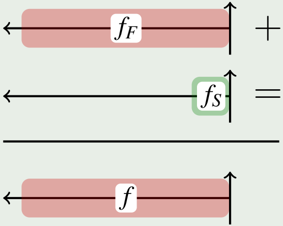

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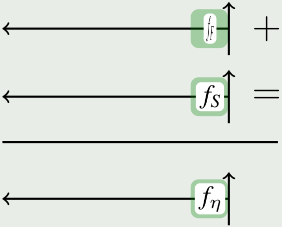

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Definition of f_η

Let $y \in \mathbb{R}^n$ and $u : [0, \eta] \to \mathbb{R}^n$ such that

$$u' = f_F(u) + f_S(y), \qquad u(0) = y.$$

Let

$$f_\eta(y) := \frac{1}{\eta}(u(\eta) - y) = f_S(y) + \frac{1}{\eta}\int_0^\eta f_F(u(s))\,\mathrm{d}s.$$

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Advantages:

- Evaluation cost of f_η is comparable to $f_F + f_S$, as f_S is frozen.
- Stiffness is reduced thanks to the smoothing effect of f_F.

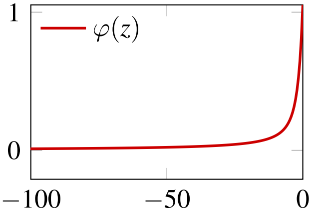
The smoothing effect

Theorem

Let $f_F(y) = A_F\, y$ *with* $A_F \in \mathbb{R}^{n\times n}$ *negative definite. Then*

$$f_\eta(y) = \varphi(\eta A_F) f(y), \qquad \textit{with} \qquad \varphi(z) = \frac{e^z - 1}{z}.$$

- $A_F < 0$ and $\varphi(z) \in (0, 1)$ for $z < 0$: $\varphi(\eta A_F)$ has a smoothing effect on $f(y)$,
- $\eta \geq 0$ is a free parameter used to tune the smoothing effect as needed.



Stiffness of the modified equation

Let the *multirate test equation* be defined by

$$y' = f_F(y) + f_S(y) = \lambda y + \zeta y, \quad \lambda, \zeta \leq 0.$$

Then

$$f_\eta(y) = \varphi(\eta\lambda)(\lambda + \zeta)y.$$

Goal: choose η such that spectrum of f_η is comparable to the one of f_S. Hence, we want $|\varphi(\eta\lambda)(\lambda+\zeta)| \leq |\zeta|$.

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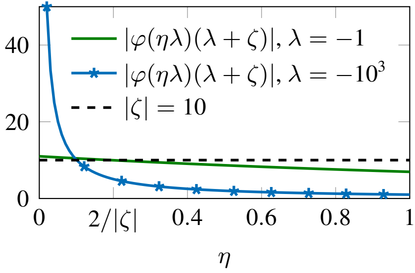
Lemma

It holds

$$|\varphi(\eta\lambda)(\lambda+\zeta)| \leq |\zeta|$$

for all $\lambda \leq 0$ if, and only if, $\eta \geq 2/|\zeta|$.



Modified multirate equation

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Solve

$$y_\eta' = f_\eta(y_\eta), \qquad y_\eta(0) = y_0$$

with

$$f_\eta(y) = \frac{1}{\eta}(u(\eta) - y),$$

where u is defined by

$$u' = f_F(u) + f_S(y), \qquad u(0) = y, \qquad \eta = 2/\rho_S$$

and ρ_S is the spectral radius of the Jacobian of f_S.

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Integrated numerically $\Longrightarrow$ different spectral properties

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where u is defined by

$$u' = f_F(u) + f_S(y), \qquad u(0) = y, \qquad \eta = ?$$

and ρ_S is the spectral radius of the Jacobian of f_S.

Multirate RKC scheme

Definition of the mRKC scheme

Let $\tau > 0$ be the step size, integrate

$$y_\eta' = \overline{f_\eta}(y_\eta), \qquad y_\eta(0) = y_0,$$

using an s-stage RKC scheme, where $\tau\rho_S \leq 2s^2$. The right-hand side $\overline{f_\eta}$ is defined by

$$\overline{f_\eta}(u_0) = \frac{1}{\eta}(u_\eta - u_0),$$

where u_η is an approximation of $u(\eta)$, solution of

$$u' = f_F(u) + f_S(u_0), \qquad u(0) = u_0, \qquad \eta = 3\tau/s^2,$$

obtained by one step of an m-stage RKC scheme, where $\eta\rho_F \leq 2m^2$.

Theorem

The scheme is stable and is first-order accurate.

Numerical experiment: diffusion through a narrow channel

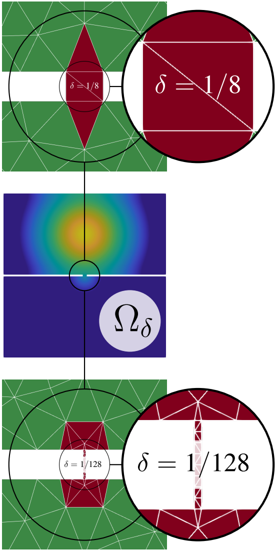
Solve

$$\begin{aligned} \partial_t u - \Delta u &= F && \text{in } \Omega_\delta \times [0,T], \\ \nabla u \cdot \boldsymbol{n} &= 0 && \text{in } \partial\Omega_\delta \times [0,T], \\ u &= 0 && \text{in } \Omega_\delta \times \{0\}, \end{aligned}$$

using first-order DG in space:

$$\begin{pmatrix} \partial_t u_h \\ \partial_t u_H \end{pmatrix} = \underbrace{\begin{pmatrix} \Delta_h u_h \\ 0 \end{pmatrix}}_{\text{Fast}} + \underbrace{\begin{pmatrix} F_h \\ \Delta_H u_H + F_H \end{pmatrix}}_{\text{Slow}}.$$

- As $\delta \to 0$ the elements in the channel get smaller.
- For varying channel width δ, compare the efficiencies of mRKC and RKC.



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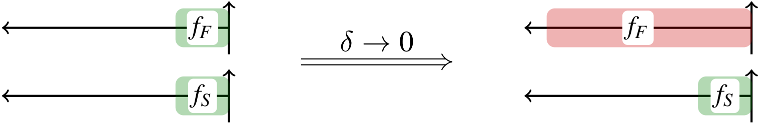

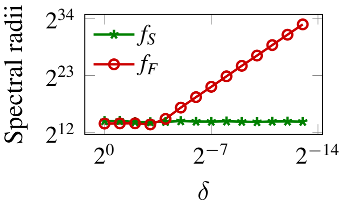

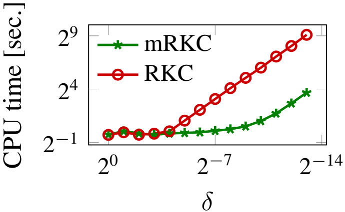

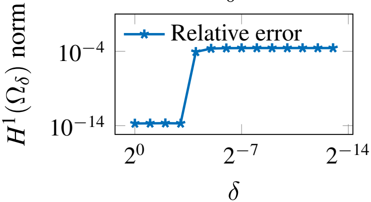

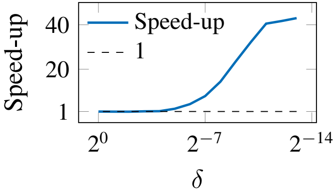

Outlook

Done:

- Extension to multirate SDEs (Abdulle and Rosilho de Souza, 2020)

$$\mathrm{d}X = f_F(X)\,\mathrm{d}t + f_S(X)\,\mathrm{d}t + g(X)\,\mathrm{d}W, \qquad X(0) = X_0.$$

Work in progress:

- Mixed precision mRKC (Croci and Rosilho de Souza, 2021):
  - accuracy steps done in *double* precision,
  - stability steps done in *half* precision.

  Preliminary results are very promising.

To do:

- Extension to multirate SDEs driven by jump-diffusion processes.

Thank you for your attention!

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