A priori and a posteriori analysis of a local scheme for elliptic equations

Assyr Abdulle, Giacomo Rosilho de Souza

ANMC, Mathematics Section, École Polytechnique Fédérale de Lausanne,
CH–1015 Lausanne, Switzerland

ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

Swiss Numerics Day: Zürich, April 20, 2018

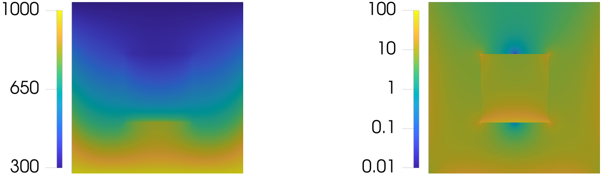
Semi linear problem:

Find $u \in H_0^1(\Omega)$ such that

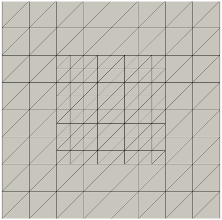
$$a(u,v) := \int_\Omega A(u)\nabla u \cdot \nabla v = \langle f, v\rangle_{H^{-1},H_0^1} \qquad \forall v \in H_0^1(\Omega),$$

with $\Omega \subset \mathbb{R}^d$, $A(u)$ symmetric and positive, $f \in H^{-1}(\Omega)$.

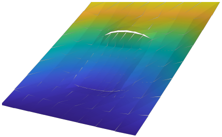
Richard's equation, with $f = 0$ and $A(u)$ discontinuous in space.



(a) Solution. (b) Norm of gradient.



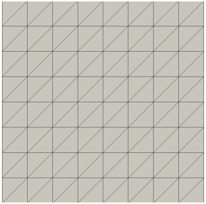
(a) Mesh $\mathcal{T}_2$.



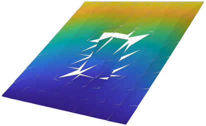
(b) Solution w_2.

Figure: Classical scheme

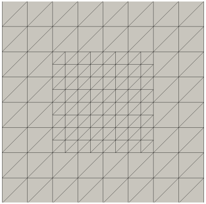
Local scheme



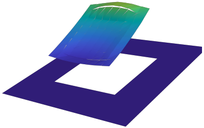
(a) First mesh $\mathcal{T}_1$.



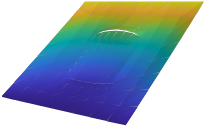
(b) First sol. u_1.



(c) Second mesh $\mathcal{T}_2$.



(d) Correction $\hat{u}_2$.



(e) Second sol. u_2.

Figure: Two iterations of the Local Scheme.

Differences and related works

Classical scheme:

- solves one non linear system on a refined mesh,
- no artificial boundary conditions error.

Local scheme:

- solves a sequence of smaller non linear systems: first on a coarse mesh, then on locally refined meshes,
- most of Newton iterations occur at coarse levels,
- artificial boundary conditions introduce additional error.

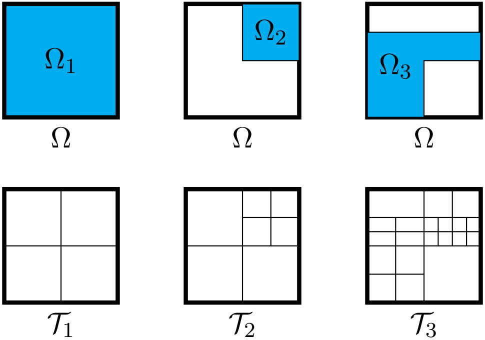
<u>Related works:</u> Methods iteratively solving a coarse full problem and a local fine problem. In finite differences framework under strong assumptions. See [Brandt, '77], [Hackbusch, '84], [McCormick, Thomas, '86].

Local scheme

Let $\Omega_1 = \Omega$, and $\Omega_k \subset \Omega$ for $k = 2, ..., M$ subdomains. Let $\{\mathcal{T}_k\}_{k=1}^{M}$ meshes on Ω and V_k discontinuous finite element spaces

$$V_k := \{v \in L^2(\Omega) \,:\, v|_T \in \mathbb{P}_d^1(T)\, \forall T \in \mathcal{T}_k\}$$

such that $V_1 \subset V_2 \subset ... \subset V_M$.



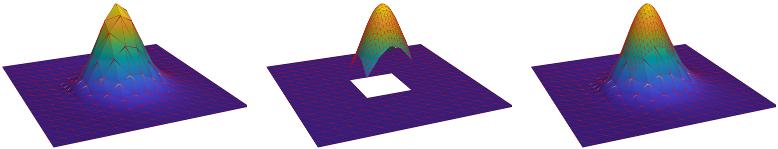
Let $a_k : V_k \times V_k \to \mathbb{R}$ be the discrete version of $a(u, v)$.

Local Scheme

Set $u_0 = 0$, for $k = 1, ..., M$ find $\hat{u}_k \in V_k$ with $\text{supp}(\hat{u}_k) \subset \Omega_k$

$$
\begin{aligned}
a_k(\hat{u}_k, v_k) &= (f, v_k)_{\Omega_k} && \forall v_k \in V_k \text{ with } \text{supp}(v_k) \subset \Omega_k \\
\hat{u}_k| &\approx u_{k-1} && \text{on } \partial\Omega_k.
\end{aligned}
$$

Set $u_k = u_{k-1}\chi_{\Omega \setminus \Omega_k} + \hat{u}_k$.



(a) First sol. u_1. (b) Correction $\hat{u}_2$. (c) Second sol. u_2.

Convergence: Semi linear and linear case

Theorem [Abdulle, Rosilho, '18]: A priori convergence

Let $u \in H^1_0(\Omega)$ be the exact solution, then for $k = 1, ..., M$

$$\lim_{h_1 \to 0} \|\nabla u - \nabla u_k\|_{L^2(\Omega)^d} + |u_k|_{J(\Omega)} = 0.$$

If $u \in H^2(\Omega) \cap H^1_0(\Omega)$, $f \in L^2(\Omega)$ and A is linear, then

$$\|\nabla u - \nabla u_k\|_{L^2(\Omega_k)^d} + |u_k|_{J(\Omega_k)} \leq C\hat{h}_k + \frac{C}{\hat{h}_k}\|v_k - u_{k-1}\|_{L^2(\partial\Omega_k)},$$

with $v_k \in V_k$ any approximation of u and $\hat{h}_k$ the local mesh size.
For $k = 2$

$$\|\nabla u - \nabla u_2\|_{L^2(\Omega_2)^d} + |u_2|_{J(\Omega_2)} \leq C\hat{h}_2 + C\hat{h}_1^{3/2}\log(1/\hat{h}_1).$$

Numerical example: Stationary Richards equation

Given a sequence of subdomains $\{\Omega_k\}_{k=1}^4$ and meshes $\{\mathcal{T}_k\}_{k=1}^4$ we compare the cost of the Classical the Local schemes.

Classical solution w_k

Cost of w_k is the solution of

$$a(w_k, v_k) = (f, v_k)_{\Omega} \quad \forall v_k \in V_k$$

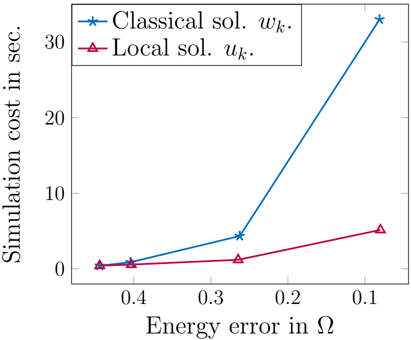
Local solution u_k

Cost of u_k is the solution of

$$a_j(\hat{u}_j, v_j) = (f, v_j)_{\Omega_j} \quad \forall v_j \in V_j$$

$$\mathrm{supp}(v_j) \subset \Omega_j$$

for $j = 1, ..., k$.



A posteriori analysis

- In practical cases it is not known a priori where the mesh has to be refined.
- We develop a posteriori error estimators which are used to define the local domains Ω_k.

Model Problem:

Find $u \in H_0^1(\Omega)$ such that

$$\int_\Omega A\nabla u \cdot \nabla v + \boldsymbol{\beta} \cdot \nabla u v + \mu u v = (f, v)_\Omega \qquad \forall v \in H_0^1(\Omega),$$

with $\Omega \subset \mathbb{R}^d$, A symmetric and positive, $\boldsymbol{\beta}$ velocity field, μ reaction and $f \in L^2(\Omega)$.

Diffusive Flux Reconstruction

A posteriori error estimators based on fluxes $\boldsymbol{t}_k \approx -A\nabla u_k$ locally in H_{div}, with jumps at the interface between sub domains.

- Start with $\boldsymbol{t}_0 = 0$, $\boldsymbol{t}_0 \in L^2(\Omega)^d$. For each k:
- Given current local solution $\hat{u}_k$, compute local flux $\hat{\boldsymbol{t}}_k \approx -A\nabla \hat{u}_k$ with $\hat{\boldsymbol{t}}_k \in H_{\text{div}}(\Omega_k)$.
- Update $\boldsymbol{t}_k = \boldsymbol{t}_{k-1}\chi_{\Omega\setminus\Omega_k} + \hat{\boldsymbol{t}}_k \notin H_{\text{div}}(\Omega)$.

Similar for a convection reconstruction $\boldsymbol{q}_k \approx \boldsymbol{\beta} u_k$.

Conservation property

Lemma [Abdulle, Rosilho, '18]: Local conservation property

Let $u_k \in V_k$ be defined by the local algorithm and $\boldsymbol{t}_k, \boldsymbol{q}_k \in L^2(\Omega)^d$ the reconstructed fluxes. For all $K \in \mathcal{T}_k$ it holds

$$\nabla \cdot \boldsymbol{t}_k + \nabla \cdot \boldsymbol{q}_k + \pi_\ell(\mu - \nabla \cdot \boldsymbol{\beta}) u_k = \pi_\ell f,$$

with π_ℓ orthogonal projector on K of order ℓ.

Theorem [Abdulle, Rosilho, '18]: Energy norm error bound

Let $u \in H_0^1(\Omega)$ be the exact solution, $u_k \in V_k$ discrete solution, then

$$\|A^{1/2}(\nabla u - \nabla u_k)\|_{L^2(\Omega)^d} + \|(\mu - \nabla\cdot\boldsymbol{\beta})^{1/2}(u - u_k)\|_{L^2(\Omega)} \leq \left(\sum_{K\in\mathcal{T}_k} \eta_K^2\right)^{1/2}$$

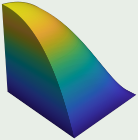
with

$$\begin{aligned}\eta_K = {} & \text{residual } |f - \nabla\cdot\boldsymbol{t}_k - \nabla\cdot\boldsymbol{q}_k - (\mu - \nabla\cdot\boldsymbol{\beta})u_k| \text{ in } K \\ & + \text{errors of approximations } \boldsymbol{t}_k \approx -A\nabla u_k \text{ and } \boldsymbol{q}_k \approx \boldsymbol{\beta} u_k \\ & + \text{jumps of } u_k \text{ on } \partial K \\ & + \text{jumps of } \boldsymbol{t}_k\cdot\boldsymbol{n}_K \text{ on } \partial K \\ & + \text{jumps of } \boldsymbol{q}_k\cdot\boldsymbol{n}_K \text{ on } \partial K.\end{aligned}$$

Numerical example: Singularly perturbed problem

Problem

Solve $-\varepsilon\Delta u + \boldsymbol{\beta}\cdot\nabla u + \mu u = f$ in $\Omega = [0,1]^2$,
with $\varepsilon = 10^{-5}$, $\boldsymbol{\beta} = -(1,1)^\top$, $\mu = 2$.



We solve the problem with the Classical and Local scheme and compare accuracy, costs, error estimators.

Classical solution w_k

Cost of w_k is the solution of

$$a_j(w_j, v_j) = (f, v_j)_\Omega \quad \forall v_j \in V_j$$

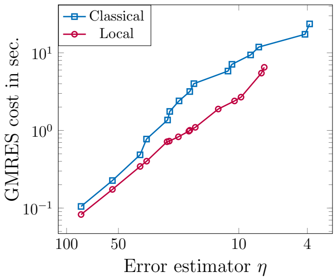
for $j = 1, ..., k$.

Local solution u_k

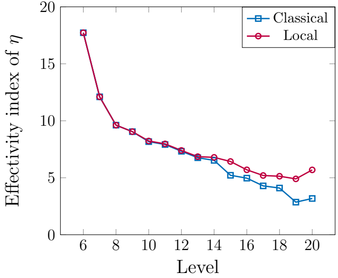
Cost of u_k is the solution of

$$a_j(\hat{u}_j, v_j) = (f, v_j)_{\Omega_j} \quad \forall v_j \in V_j$$
$$\text{supp}(v_j) \subset \Omega_j$$

for $j = 1, ..., k$.

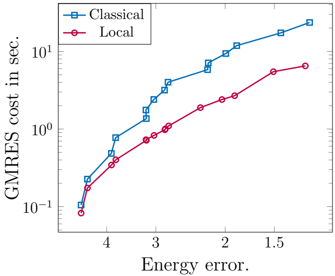


(a) η versus GMRES cost.

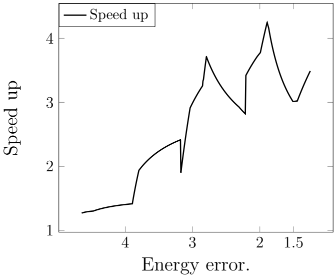


(b) Effectivity index of η.

Figure: Estimator η versus cost and its effectivity index.



(a) Energy error versus GMRES cost.



(b) Speed up in function of the error

Figure: Error versus cost and speed up in function of the error.

Thanks for your attention!